MATHEMATICAL METHODS II

Learning Outcomes
At the end of the course, students should be able to:
1. Describe Sturm-Liouville problems.
2. Discuss orthogonal polynomials and functions.
3. Solve some problems using first and second order differential equations.

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1. Introduction to Sturm-Liouville Theory

Sturm-Liouville theory is a branch of mathematics that deals with a special type of second-order linear differential equation of the form:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) + \lambda r(x)u(x) = 0$$

where $p(x)$, $q(x)$, and $r(x)$ are continuous functions on a closed interval $[a, b]$ and λ is a constant parameter.

These differential equations are named after the French mathematician Jacques Charles François Sturm and the French mathematician Joseph Liouville, who developed this theory in the mid-19th century. The solutions to such equations have important applications in fields such as physics, engineering, and finance. In Sturm-Liouville theory, the focus is on finding the eigenvalues and eigenfunctions of the differential equation.

Sturm-Liouville theory is used in various fields of mathematics and science, including partial differential equations, Fourier analysis, quantum mechanics, and more. It is an essential tool in many areas of research and is still an active area of study.

1.1 Inner Products with Weight Functions

An inner product is a mathematical operation that takes two vectors and returns a scalar. It is also known as a *dot product* or *scalar product*. The result of an inner product is the projection of one vector onto another vector, scaled by the magnitude of the second vector. The inner product has many applications in mathematics, physics, and engineering. For example, it is used to calculate the angle between two vectors, the length of a vector, and the distance between two points in space. It is also used in linear algebra to define orthogonal projections, orthogonality, and the Gram-Schmidt process.

The inner product of two vectors x and y is defined as follows:

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are the components of vectors x and y, respectively.

Suppose that $w(x)$ is a nonnegative function on $[a, b]$. If $f(x)$ and $g(x)$ are real-valued functions on $[a, b]$ we define their inner product on $[a, b]$ with respect to the weight w to be;

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$$

We say f and g are orthogonal on $[a, b]$ with respect to the weight w if;

$$\langle f, g\rangle = 0$$

Remarks:

- The inner product and orthogonality depend on the choice of a, b and w.
- When $w(x) \equiv 1$, these definitions reduce to the "ordinary" ones.

Examples:

- The functions $f_n(x) = \sin(nx)$, $(n = 1, 2, \ldots)$ are pairwise orthogonal on $[0, \pi]$ relative to the weight function $w(x) \equiv 1$.
- Let J_m be the Bessel function of the first kind of order m, and let α_{mn} denote its nth positive zero. Then the functions;
$$f_n(x) = J_m(\alpha_{mn} x/a)$$
are pairwise orthogonal on $[0, a]$ with respect to the weight function $w(x) = x$.
- The functions;
$$f_0(x) = 1, \qquad f_1(x) = 2x, \ \ f_2(x) = 4x^2 - 1, \ \ f_3(x) = 8x^3 - 4x,$$
$$f_4(x) = 16x^4 - 12x^2 + 1, \qquad f_5(x) = 32x^5 - 32x^3 + 6x$$
are pairwise orthogonal on $[-1, 1]$ relative to the weight function $w(x) = \sqrt{(1 - x^2)}$. They are examples of Chebyshev polynomials of the second kind.

1.2 Series Expansions

We have frequently seen the need to express a given function as a linear combination of an orthogonal set of functions. Our fundamental result generalizes to weighted inner products.

Theorem 1.1: Suppose that $\{f_1, f_2, f_3, \ldots\}$ is an orthogonal set of functions on $[a, b]$ with respect to the weight function w. If f is a function on $[a, b]$ and
$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$
then the coefficients a_n are given by;
$$a_n = \frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{\int_a^b f(x) f_n(x) w(x) dx}{\int_a^b f_n^2(x) w(x) dx}$$

Remarks:

- The series expansion above is called a generalized Fourier series for f, and a_n are the generalized Fourier coefficients.
- It is natural to ask:
  - Where do orthogonal sets of functions come from?
  - To what extent is an orthogonal set complete, i.e. which functions f have generalized Fourier series expansions?
- In the context of PDEs, these questions are answered by Sturm-Liouville Theory

1.3 Sturm-Liouville Equations

A Sturm-Liouville equation is a second order linear differential equation that can be written in the form;
$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$
Such an equation is said to be in Sturm-Liouville form.

- Here p, q and r are specific functions, and λ is a parameter.
- Because λ is a parameter, it is frequently replaced by other variables or expressions.
- Many "familiar" ODEs that occur during separation of variables can be put in Sturm-Liouville form.

Example: Show that $y'' + \lambda y = 0$ is a Sturm-Liouville equation.

We simply take $p(x) = r(x) = 1$ and $q(x) = 0$

Example: Put the parametric Bessel equation;

$$x^2y'' + xy' + (\lambda^2x^2 - m^2)y = 0$$

in Sturm-Liouville form.

First, we divide by x to get;

$$xy'' + y' + \left(\lambda^2x - \frac{m^2}{x}\right)y = 0, \text{ where } (xy')' = xy'' + y'$$

This is in Sturm-Liouville form with;

$$p(x) = x, \ q(x) = -\frac{m^2}{x}, \ r(x) = x$$

provided we write the parameter as λ^2.

Example: Put Legendre's differential equation;

$$y'' - \frac{2x}{1-x^2}y' + \frac{\mu}{1-x^2}y = 0$$

in Sturm-Liouville form.

Solution:

First, we multiply by $1 - x^2$ to get;

$$(1-x^2)y'' - 2xy' + \mu y = 0, \text{ where } ((1-x^2)y')'$$

is in Sturm-Liouville form with;

$$p(x) = 1 - x^2, \ q(x) = 0, \ r(x) = 1$$

provided we write the parameter as μ.

Example: Put Chebyshev's differential equation;

$$(1-x^2)y'' - xy' + n^2y = 0$$

in Sturm-Liouville form.

Solution:

First, we divide by $\sqrt{(1-x^2)}$ to get;

$$\sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{n^2}{\sqrt{1-x^2}}y = 0, \text{ where } \left(\sqrt{1-x^2y'}\right)' = \sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y'$$

This is in Sturm-Liouville form with;

$$p(x) = \sqrt{1-x^2}, \ q(x) = 0, \ r(x) = \frac{1}{\sqrt{1-x^2}}$$

provided we write the parameter as n^n.

1.4 Sturm-Liouville Problems

A Sturm-Liouville problem consists of a Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \ a < x < b \quad (1)$$

together with boundary conditions, i.e. specified behaviour of y at $x = a$ and $x = b$.

We will assume that p, p', q and r are continuous and $p > 0$ on (at least) the open interval $a < x < b$. According to the general theory of second order linear ODEs, this guarantees that solutions to (1) exist.

1.4.1 Regularity conditions

A regular Sturm-Liouville problem has the form;

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b$$

$$c_1 y(a) + c_2 y'(a) = 0 \tag{2}$$

$$d_1 y(b) + d_2 y'(b) = 0 \tag{3}$$

where;

- $(c_1, c_2) \neq (0, 0)$ and $(d_1, d_2) \neq (0, 0)$
- p, p', q and r are continuous on $[a, b]$

The boundary conditions (2) and (3) are called separated boundary conditions.

Example: The boundary value problem;

$$y'' + \lambda y = 0, \quad 0 < x < L$$

$$\text{BC: } y(0) = y(L) = 0$$

is a regular Sturm-Liouville problem (recall that $p(x) = r(x) = 1$ and $q(x) = 0$).

Example: The boundary value problem;

$$\left((x^2 + 1)y'\right)' + (x + \lambda)y = 0, \quad -1 < x < 1$$

$$\text{BC: } y(-1) = y'(1) = 0$$

is a regular Sturm-Liouville problem (here $p(x) = x^2 + 1$, $q(x) = x$ and $r(x) = 1$).

Example: The boundary value problem;

$$x^2 y'' + xy' + (\lambda^2 x^2 - m^2)y = 0, \quad 0 < x < a$$

$$y(a) = 0$$

is not a regular Sturm-Liouville problem.

Why not? Recall that when put in Sturm-Liouville form, we had;

$$p(x) = r(x) = x \text{ and } q(x) = -m^2/x.$$

There are several problems:

- p and r are not positive when $x = 0$
- q is not continuous when $x = 0$
- The boundary condition at $x = 0$ is missing

This is an example of a singular Sturm-Liouville problem.

1.5 Eigenvalues and eigenfunctions

A nonzero function y that solves the Sturm-Liouville problem;

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b \qquad \text{(plus boundary conditions)}$$

is called an eigenfunction, and the corresponding value of λ is called its eigenvalue:

- The eigenvalues of a Sturm-Liouville problem are the values of λ for which nonzero solutions exist.
- We can talk about eigenvalues and eigenfunctions for regular or singular problems.

Example: Find the eigenvalues of the regular Sturm-Liouville problem;

$$y'' + \lambda y = 0, \quad 0 < x < L$$
$$y(0) = y(L) = 0$$

This problem first arose when we separated variables in the 1-D wave equation. We already know that nonzero solutions occur only when;

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2} \text{ (eigenvalues)}$$

and

$$y = y_n = \sin\frac{n\pi x}{L} \qquad \text{(eigenfunctions)}$$

for $n = 1, 2, 3, \ldots$

Example: Find the eigenvalues of the regular Sturm-Liouville problem;

$$y'' + \lambda y = 0, \quad 0 < x < L$$

$y(0) = 0, \quad y(L) + y'(L) = 0$

This problem arose when we separated variables in the 1-D heat equation with *Robin conditions*. We already know that nonzero solutions occur only when;

$$\lambda = \lambda_n = \mu_n^2$$

where μ_n is the nth positive solution to:

$$\tan \mu L = -\mu$$

and

$$y = y_n = \sin(\mu_n x)$$

for $n = 1, 2, 3, \ldots$

Example: If $m \geq 0$, find the eigenvalues of the singular Sturm-Liouville problem;

$$x^2y'' + xy' + (\lambda^2x^2 - m^2)y = 0, \quad 0 < x < a$$

$y(0)$ is finite, $\quad y(a) = 0$

This problem arose when we separated variables in the vibrating circular membrane problem. We know that nonzero solutions occur only when;

$$\lambda = \lambda_n = \frac{\alpha_{mn}}{a}$$

where α_{mn} is the nth positive zero of the Bessel function J_m, and;

$$y = y_n = J_m(\lambda_n x)$$

for $n = 1, 2, 3, \ldots$ (technically, the eigenvalues are $\lambda_n^2 = a_{mn}^2/a^2$).

The previous examples demonstrate the following general properties of a regular Sturm-Liouville problem;

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \; a < x < b$$

$c_1y(a) + c_2y'(a) = 0, \quad d_1y(b) + d_2y'(b) = 0$

Theorem 1.1 The eigenvalues form an increasing sequence of real numbers;

$$\lambda_{1,} < \lambda_2 < \lambda_3 < \cdots$$

with

$$\lim_{n\to\infty} \lambda_n = \infty$$

Moreover, the eigenfunction $y_n$corresponding to λ_n is unique (up to a scalar multiple), and has exactly $n - 1$ zeros in the interval $a < x < b$.

Another general property is the following:

Theorem 1.2 Suppose that $y_j$and y_k are eigenfunctions corresponding to distinct eigenvalues λ_j and λ_k . Then $y_j$and y_k are orthogonal on $[a, b]$ with respect to the weight function $w(x) = r(x)$. That is;

$$\langle y_j, y_k \rangle = \int_a^b y_j(x) y_k(x) r(x) dx = 0$$

- This theorem actually holds for certain non-regular Sturm-Liouville problems, such as those involving Bessel's equation.
- Applying this result in the examples above we immediately recover familiar orthogonality statements.
- This result explains why orthogonality figures so prominently.

Example: Write down the conclusion of the orthogonality theorem for;

$$y'' + \lambda y = 0, \quad 0 < x < L$$

$y(0) = y(L) = 0$

Since the eigenfunctions of this regular Sturm-Liouville problem are;

$$y_n = \sin\left(\frac{n\pi x}{L}\right), \qquad \text{and} \quad \text{since } r(x) = 1,$$

we immediately deduce that;

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

for $m \neq n$.

Example: If $m \geq 0$, write down the conclusion of the orthogonality theorem for;

$$x^2 y'' + x y' + (\lambda^2 x^2 - m^2) y = 0, \quad 0 < x < a$$

$y(0)$ is finite, $\quad y(a) = 0$.

Since the eigenfunctions of this regular Sturm-Liouville problem are;

$$y_n = J_m(\alpha_{mn} x / a)$$

and since $r(x) = x$, we immediately deduce that;

$$\int_0^a J_m\left(\frac{\alpha_{mk}}{a} x\right) J_m\left(\frac{\alpha_{ml}}{a} x\right) dx = 0$$

for $k \neq l$.

2. Orthogonal Polynomials

The study of orthogonal polynomials dates back to the 18th century, with the works of mathematicians such as Leonard Euler and Joseph-Louis Lagrange. However, it was not until the 19th century that the concept of orthogonal polynomials was fully developed by mathematicians such as Gustav Jacobi and Charles Hermite.

One of the early motivations for the study of orthogonal polynomials was the need to solve problems in calculus, particularly in the area of numerical integration. By finding polynomials that were orthogonal with respect to a given weight function, mathematicians were able to construct numerical integration formulas that were more accurate than previous methods. The study of orthogonal polynomials also found applications in physics, particularly in the areas of quantum mechanics and statistical mechanics. The orthogonality properties of certain families of polynomials allowed physicists to develop models of quantum systems and to calculate various physical quantities, such as energy levels and transition probabilities.

Today, the study of orthogonal polynomials continues to be an active area of research in mathematics, with applications in areas such as approximation theory, number theory, and combinatorics. The development of new families of orthogonal polynomials with different properties and applications remains an active area of research.

2.1 What is Orthogonal Polynomials?

Orthogonal polynomials are a set of polynomials that satisfy a specific orthogonality condition with respect to a given weight function over a certain interval. The orthogonality condition requires that the integral of the product of two different polynomials, each multiplied by the weight function, is equal to zero, except when the polynomials are identical.

Orthogonal polynomials can be used to approximate a function over a certain interval using a linear combination of the polynomials. This technique is known as polynomial interpolation and has applications in numerical analysis and computer graphics. These polynomials have important applications in various fields, including mathematics, physics, engineering, and computer science. Examples of orthogonal polynomials include the Legendre, Hermite, Chebyshev, Jacobi polynomials, and Laguerre polynomials.

Definition 2.1: A polynomial is a function p whose value at x is;

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \dots, a_2, a_1$ and a_0, called the coefficients of the polynomial, are constants and , if $n > 0$, then $a_n \neq 0$. The number n, the degree of the highest power of x in the polynomial, is called the degree of the polynomial – the degree of the zero polynomial is not defined.

Definition 2.2: Let $w(x)$ is a weight function and $p_n(x)$ polynomials are defined over the interval $[a, b]$, if;

$$\int_a^b p_n(x) p_m(x) w(x) dx = 0, \quad m \neq n \tag{2.1}$$

is satisfied, then the polynomials $p_n(x)$ are called orthogonal with respect to the weight function $w(x)$ over the interval (a, b); m and n are degrees of the polynomials.

There is an additional condition for the orthogonal polynomials which makes them **orthonormal**:

Definition 2.3: If the polynomials $p_n(x)$ are orthogonal with respect to the weight function $w(x)$, over the interval (a, b) and;

$$\|p_n(x)\|^2 = \int_a^b \int_a^b p_n^2(x)w(x)dx = 1, \qquad m = n$$

is satisfied , then the polynomials $p_n(x)$ are called **orthonormal.**

There is an equivalent condition for the orthogonality relation (2.1) which is given below:

Theorem 2.4: It is sufficient for the orthogonality of the polynomials on the interval $[a, b]$ with respect to the weight function $w(x)$ to satisfy the condition:

$$\int_a^b \varphi_n(x)x^i w(x)dx = 0, \qquad i = 0,1,2, \dots, n-1 \tag{2.2}$$

here, $\varphi_n(x)$ is polynomial of degree n.

Proof.

If the polynomials $\varphi_n(x)$ and $\varphi_m(x)$ are orthogonal on the interval $[a, b]$ with respect to $w(x)$ then;

$$\int_a^b \varphi_n(x)x^i w(x)dx = 0, \qquad m \neq n$$

x^i, can be written as linear combinations,

$$x^i = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + \dots + a_i\varphi_i = \sum_{m=1}^{i} a_m\varphi_m(x)$$

substituting this in (2.2);

$$\int_a^b \varphi_n(x)x^i w(x)dx = \int_a^b w(x)\varphi_n(x)\left\{\sum_{m=0}^{i} a_m\varphi_m(x)\right\}dx$$

$$= \sum_{m=0}^{i} a_m \int_a^b w(x)\varphi_n(x)\varphi_m(x)dx = 0$$

for $0 \leq m \leq i,\ \varphi_n(x)$ and $\varphi_m(x)$ where $0 \leq m < n$.

Hence;

$$\int_a^b \varphi_n(x)x^i w(x)dx = 0, \qquad i = 0,1,2, \dots, n-1$$

Orthogonal polynomials have several important properties. In this section, general definitions of these properties are given and then obtained special form of them for well-known orthogonal polynomial families. We shall state the following definitions without proof.

Definition 2.5: Any polynomial family $\varphi_n(x)$, which is orthogonal on the interval $[a, b]$ with respect to the weight function $w(x)$, satisfies the recurrence formula:

$$\varphi_{n+1}(x) - (xA_n + B_n)\varphi_n(x) + C_n\varphi_{n-1}(x) = 0$$

here, A_n, B_n and C_n are constants which depend on n.

Definition 2.6: Rodrigues formula for orthogonal polynomials are written as;

$$\varphi_n(x) = A_n \frac{1}{w(x)} \frac{d^n}{dx^n}[w(x)u^n(x)], \qquad n = 0,1,2,\dots \tag{2.3}$$

here, polynomials are orthogonal with respect to the weight function $w(x)$ and u^n is a polynomial of x.

Definition 2.7: If the two variable function $F(x,t)$ has a Taylor series as in the form of:

$$F(x,t) = \sum_{n=0}^{i} a_n \varphi_n(x) t^n \tag{2.4}$$

with respect to one of its variables t, then the function $F(x,t)$ is called the *generating function* for the polynomials $\{\varphi_n(x)\}$.

Definition 2.8 (Bilateral Generating Function): If the three variable function $H(x,y,t)$ has a Taylor series in the form of:

$$H(x,y,t) = \sum_{n=0}^{\infty} C_n f_n(x) g_n(x) t^n$$

with respect to one of its variables, t, then the function $H(x,y,t)$ is the bilateral generating function for the families f_n and g_n.

Definition 2.9 (Bilinear Generating Function): If the three variable function $G(x,y,t)$ has a Taylor series in the form of:

$$G(x,y,t) = \sum_{n=0}^{\infty} c_n f_n(x) g_n(y) t^n$$

with respect to one of its variables , t , then the function $G(x,y,t)$ is the bilinear generating function for the families function f_n and g_n.

2.2 Some Special Orthogonal Polynomial Families

Some well-known orthogonal polynomials families which have several applications in applied mathematics are given at this section. These polynomial families have several properties which are common and obtainable for any orthogonal polynomial family.

2.2.1 Jacobi Polynomials

Jacobi polynomials are a family of orthogonal polynomials that are named after the German mathematician Carl Gustav Jacob Jacobi. They arise in various areas of mathematics and physics, such as in numerical analysis, approximation theory, and quantum mechanics.

The Jacobi polynomials have many interesting properties and applications, including the representation of functions as series of Jacobi polynomials, the solution of certain differential equations, and the approximation of functions by polynomial interpolation.

For $\alpha > -1, \beta > -1$, the Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$, which is orthogonal on the interval $-1 \leq x \leq 1$ with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$, are given by the formula;

$$p_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x+1)^k (x-1)^{n-k}, \quad n = 1,2,\dots$$

If $\alpha = \beta$, the polynomials $p_n^{(\alpha,\beta)}(x)$ are called "**Ultraspherical Polynomials**".

Some special cases of Jacobi polynomials which depend on the values of α and β are given below:

1. For $\alpha = \beta = -\frac{1}{2}$, the polynomials;

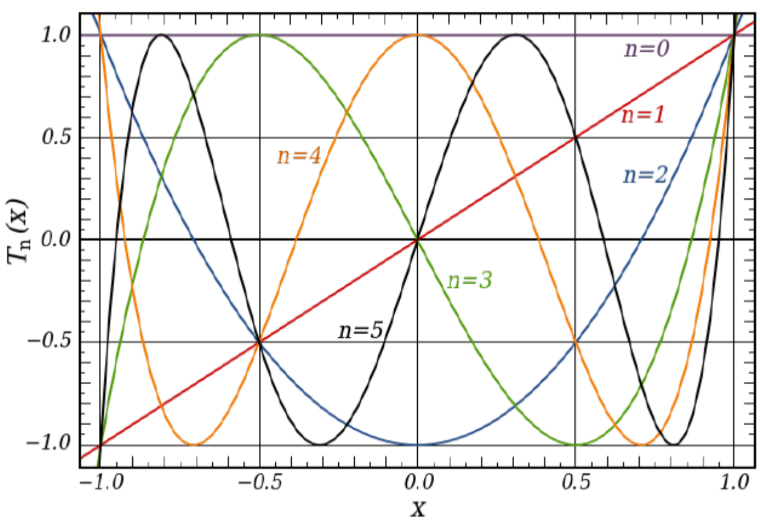
$$p_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!\, x^{n-2k}(x^2-1)^k}{(2k)!\,(n-2k)!} = T_n(x)$$

are called "**I. Type Chebyshev Polynomials**".

Some of the polynomials $T_n(x)$ are;

$T_0(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$

The graphs of first six I. Type Chebyshev Polynomials $T_0(x), T_1(x), T_2(x), T_3(x), T_4(x)$ and $T_5(x)$ are shown in the figure:



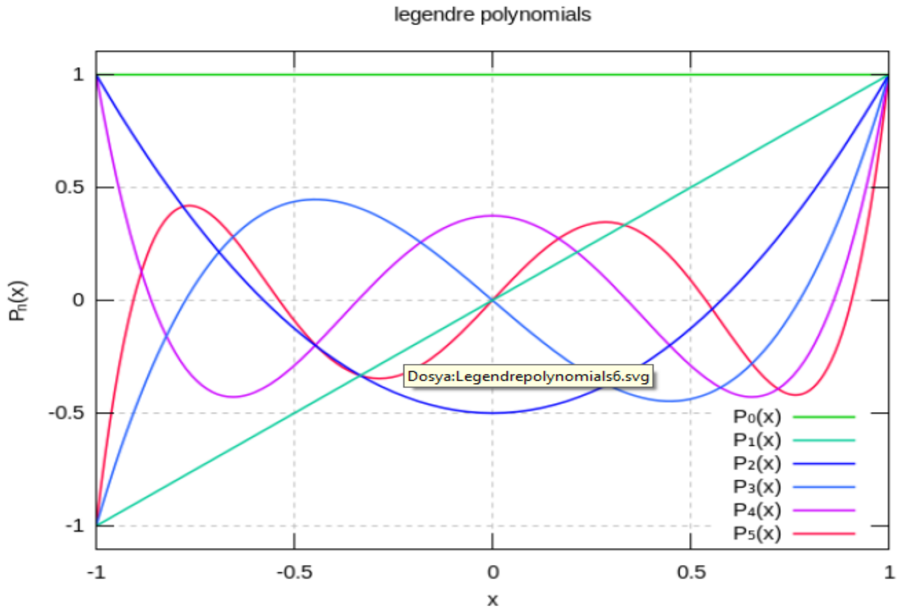
2. For $\alpha = \beta = 0$, the polynomials;

$$p_n^{(0,0)}(x) = 2^{-n} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} = p_n(x)$$

are called " **Legendre Polynomials**". Let us give the first five Legendre polynomials:

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$
The graphs of first six Legendre polynomials $P_0(x), P_1(x), P_3(x), P_4(x)$ and $P_5(x)$ are shown in the figure:



Here

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

If

$$\frac{d}{dx}\left[(1-x^2)(1-x)^{\alpha}(1+x)^{\beta}\frac{d}{dx}P_n^{(\alpha,\beta)}(x)\right]$$

is used to start, the Jabobi differential equation can be obtained as;

$$(1-x^2)y'' + [\beta - \alpha(\alpha+\beta+2)x]y + n(n+\beta+\alpha+1)y = 0$$

which has the solutions as Jacobi polynomials.

Generating function for the Jacobi polynomials are given as;

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = \frac{2^{\alpha+\beta}}{\sqrt{1-2tx+t^2}\,\left[1-t+\sqrt{1-2xt+t^2}\right]\left[1+t+\sqrt{1-2xt+t^2}\right]}$$

Finally, the recurrence relation for Jacobi polynomials are given as;

$$\begin{aligned} 2(n+1)(n+\alpha+\beta-1)(2n+\beta+\alpha)P_{n+1}^{(\alpha,\beta)} \\ - [(2n+\alpha+\beta+1)(\alpha^2-\beta^2)(2n+\alpha\beta+\beta)x]P_n^{(\alpha,\beta)}(x) \\ + 2(n+\alpha)(\alpha+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x) = 0 \end{aligned}$$

2.2.2 Hermite Polynomials

The $H_n(x)$ Hermit polynomials, which are orthogonal on the interval $-\infty < x < \infty$ with respect to the weight function $w(x) = e^{-x^2}$ given by;

$$\varphi_n(x) = H_n(x) = \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k n!}{k!\,(n-2k)!}(2x)^{n-2k}, \quad n = 0,1,2,\dots$$

Some of the polynomials $H_n(x)$ are;

$H_0(x) = 1$

$H_1(x) = 2x$

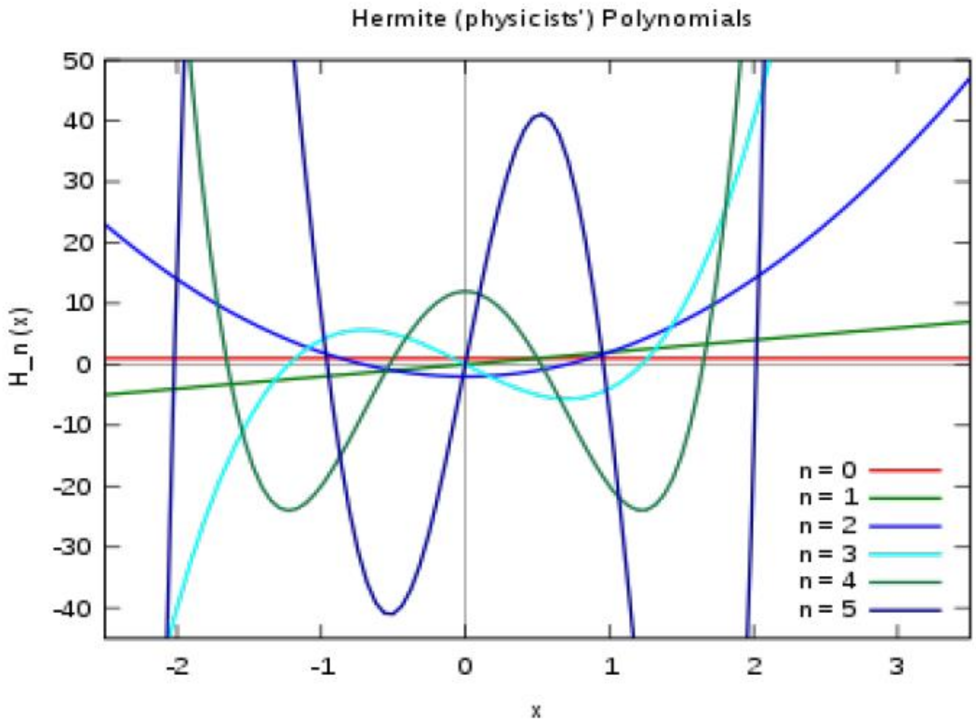
$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^3 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

The graphs of first six Hermit polynomials and are shown in the figure: $H_0(x), H_1(x), H_2(x), H_3(x), H_4(x)$ and $H_5(x)$ are shown in the figure:



Rodrigues formula for Hermite polynomials is given as:

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

The generating function for the Hermite polynomials:

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad (2.5)$$

Norm of the Hermite polynomials:

$$\|H_n(x)\|^2 = \int_{-\infty}^{\infty} e^{-x^2} H_n(x)dx = 2^n \sqrt{\pi}\, n!$$

Form the equation;

$$\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}H_n(x)\right]$$

The Hermite differential equation can be obtained as;

$$y'' - 2xy + 2ny = 0$$

Which has the solution as Hermite polynomials.

Finally, the recurrence relation for the Hermite polynomials is given as;

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \qquad (2.6)$$

By using generating function, (2.5), we can obtain the recurrence relation above by following the steps: Take the derivative of both sides in (2.7) with respect to t;

$$(2x-2t)e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}nt^{n-1}$$

$$(2x-2t)\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n = \sum_{n=0}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1}$$

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n - \sum_{\substack{n=0\\ n\to n-1}}^{\infty}\frac{2H_n(x)}{n!}t^{n+1} = \sum_{n=0}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1}$$

if the indices are manipulated to make all powers of t as t^n;

$$\sum_{n=0}^{\infty}\frac{2H_n(x)}{n!}t^n - \sum_{n=0}^{\infty}\frac{2H_{n-1}(x)}{(n-1)!}t^n = \sum_{n=0}^{\infty}\frac{H_{n+1}(x)}{n!}t^n$$

and open some terms to start the summations from 1.

$$2xH_0(x) - \sum_{n=1}^{\infty}(2xH_n(x) - 2nH_{n-1}(x)\frac{t^n}{n!} = H_0(x) + \sum_{n=1}^{\infty}H_{n+1}(x)\frac{t^n}{n!}$$

is obtained. By the equality of the coefficients of the term $\frac{t^n}{n!}$;

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)$$

can be written, which gives the recurrence relation (2.6).

2.2.3 Laguerre Polynomials

For $\alpha > -1$, the $L_n^{(\alpha)}(x)$ polynomials, which are orthogonal on $0 \le x < \infty$ with respect to the weight function $w(x) = x^\alpha e^{-x}$ and which are known as Laguerre polynomials are given by;

$$\varphi_n(x) = L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \quad n = 0,1,2,\ldots$$

The special case is $\alpha = 0$ is $L_n^{(\alpha)}(x) = L_n(x)$. Let us give the first five Laguerre polynomials;

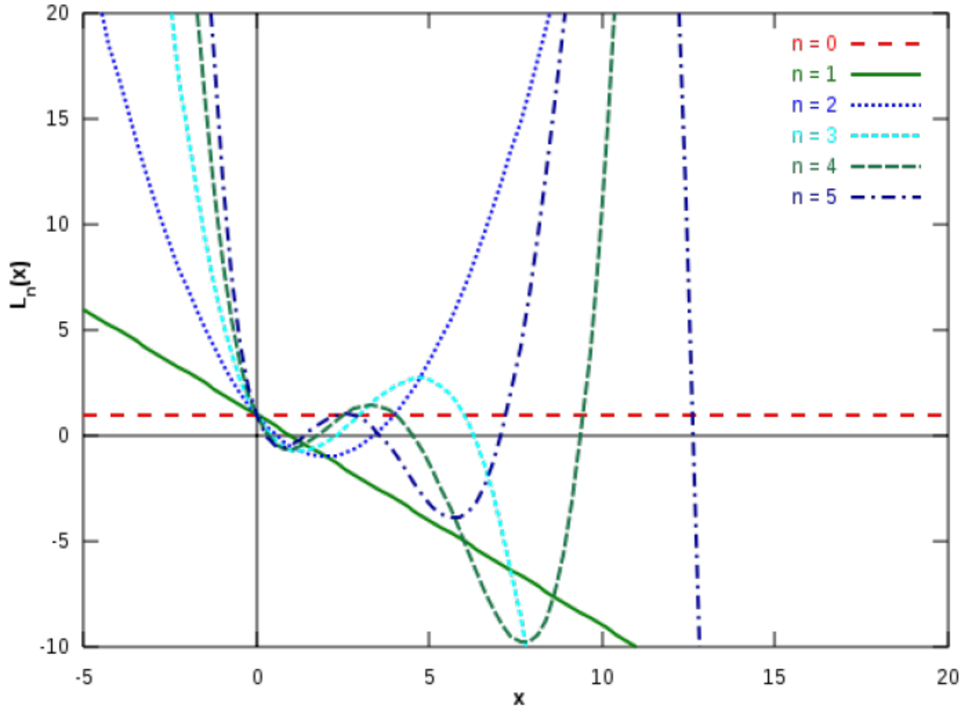
$L_0(x) = 1$

$L_1(x) = -x + 1$

$L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$

$$L_3(x) = \frac{1}{6}(-x^3 - 16x^2 - 18x + 6)$$
$$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$
$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$$

The graphs of first six Laguerre polynomials $L_0(x), L_1(x), L_3(x), L_4(x)$ and $L_5(x)$ are shown in the figure:



Several properties of Laguerre polynomials similar to orthogonal polynomials can be obtained. One of these properties is that it satisfies second order differential equations. Starting from;

$$\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right]$$

we obtain Leguerre differential equation;

$$xy'' + (\alpha + 1 - x)y' + ny = 0$$

where the solution of this differential equation are Laguerre polynomials and can be obtained.

Let us start with the equation below:

$$\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right] = x^{\alpha}e^{-x}\left[x\frac{d^2}{dx^2}L_n(x) + (\alpha + 1 - x)\frac{d}{dx}L_n(x)\right]$$

it can be written as linear combinations;

$$x\frac{d^2}{dx^2}L_n(x) + (\alpha + 1 - x)\frac{d}{dx}L_n(x) = \sum_{i=1}^{n} \alpha_i L_i(x)$$

Therefore,

$$\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right] = x^{\alpha}e^{-x}\sum_{i=1}^{n} \alpha_i L_i(x)$$

by integrating over the interval $(0,\infty)$ is deduced that;

$$\int_0^\infty L_j(x)\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right]dx = \int_0^\infty L_j(x)x^\alpha e^{-x}\sum_{i=1}^{n}\alpha_i L_i(x)$$

$$= \alpha_j\int_0^\infty x^\alpha e^{-x}L_j^2(x)dx + \sum_{\substack{i=1\\j=1\\i\neq j}}^{n}\alpha_i\int_0^\infty e^{-x}x^\alpha L_j(x)L_i(x)dx$$

it is known that the Laguerre polynomial are orthogonal, then;

$$\int_0^\infty e^{-x}x^\alpha L_j(x)L_i(x)dx = 0, \quad i\neq j$$

Consequently,

$$\int_0^\infty L_j(x)\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right]dx = \alpha_i\int_0^\infty x^\alpha e^{-x}L_j^2(x)dx + 0$$

$$\alpha_j = \frac{\int_0^\infty L_j(x)\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{d}{dx}L_n(x)\right]dx}{\int_0^\infty x^\alpha e^{-x}L_j^2(x)dx}$$

and

$$\frac{d}{dx}L_n(x) = y', \qquad \frac{d^2}{dx^2}L_n(x) = y''$$

so,

$$xy'' + (\alpha+1-x)y' + \lambda y = 0$$

$$\lambda_n = -n\left(\frac{n-1}{2}(x)'' + \alpha + 1 - x)'\right) = n$$

Then the following differential equation is obtained.

$$xy'' + (\alpha+1-x)y' + ny = 0$$

The generating function for the Laguerre polynomials;

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \frac{1}{(1-t)}\exp\left(\frac{-xt}{1-t}\right) \tag{2.7}$$

can be written. For obtaining the $\left\|L_n^{(\alpha)}(x)\right\|$ norm of Laguerre polynomials, the generating function (2.7) is rewritten as in the form of;

$$\sum_{n=0}^{\infty} e^{-x}L_m^{(\alpha)}(x)t^m = e^{-x}\frac{1}{(1-t)}\exp\left(\frac{-xt}{1-t}\right) \tag{2.8}$$

by multiplying both sides of (2.7) by $w(x) = e^{-x}$ where $m\neq n$. If (2.7 and (2.8) are multiplied side by side and integrated over the interval $(0,\infty)$;

$$\sum_{n,m=0}^{\infty}\left[\int_0^\infty e^{-x}L_n^{(\alpha)}(x)dx\right]t^{n+m} = \frac{1}{(1-t)^2}\int_0^\infty \exp\left(\frac{x(1+t)}{t-1}\right)$$

is obtained. If left hand side of the last equation is separated for $m=n$ and $m\neq n$, and take the integral at right hand side;

$$\sum_{n,m=0}^{\infty}\left[\int_0^{\infty} e^{-x}L_n^2(x)dx\right]t^{2n}+\sum_{n,m=0}^{\infty}\int_0^{\infty}\left[e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx\right]t^{n+m}=\frac{1}{(1-t)^2}\cdot\frac{1-t}{(1+t)}=\frac{1}{1-t^2}$$

is obtained. By using the orthoganality of Laguerre polynomials, for $n=m$, second integral at the left hand side is equal to zero.

If the Taylor series;

$$\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n$$

is used on the right hand side of the last equality, then;

$$\sum_{n,m=0}^{\infty}\left[\int_0^{\infty} e^{-x}L_n^2(x)dx\right]t^{2n}=\sum_{n=0}^{\infty}t^{2n}$$

is obtained. Thus, equality of the coefficient of t^{2n} in both sides give the norm of Laguerre polynomials as;

$$\left\|L_n^{(\alpha)}(x)\right\|^2=\int_0^{\infty} e^{-x}L_n^2(x)dx=1$$

Finally, the recurrence relation for Laguerre polynomial $L_n^{(\alpha)}(x)$ is given as;

$$(n+1)L_{n+1}^{(\alpha)}(x)+(x-2n-1-\alpha)L_n^{(\alpha)}+(n+\alpha)L_{n-1}^{(\alpha)}(x)=0$$

3. Fourier Series

When the French mathematician Joseph Fourier (1768{1830) was trying to solve a problem in heat conduction, he needed to express a function as an infinite series of sine and cosine functions:

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad (3.1)$$

Earlier, Daniel Bernoulli and Leonard Euler had used such series while investigating problems concerning vibrating strings and astronomy. The series in Equation 3.1 is called a trigonometric series or Fourier series and it turns out that expressing a function as a Fourier series is sometimes more advantageous than expanding it as a power series. In particular, astronomical phenomena are usually periodic, as are heartbeats, tides, and vibrating strings, so it makes sense to express them in terms of periodic functions.

Thus, the subject of Fourier series is concerned with functions on $\mathbb{R}$ that are periodic, or equivalently, are defined on a bounded interval $[a,b)$ which may then be extended periodically to all of $\mathbb{R}$. A function f on $\mathbb{R}$ is periodic with period T if ;

$$f(t+T)=f(t) \quad \forall\, t$$

(and conventionally we take the smallest such T). Thus f is fully specified if we give its values only on $[0,T)$ or any other interval of length at least T.

Geometrically, a function f is periodic with period T if the graph of f is invariant under translation in the x-direction by a distance of T.

Some definitions on periodic functions:

$$f(t)=A\sin\omega t \text{ and } g(t)=A\cos\omega t$$

here, A is the **amplitude**. Interpreting the variable t, as time, we have;

$$\textbf{period}, T = \frac{2\pi}{\omega} = \text{time iterval of a single wave}$$

$$\textbf{frequency}, f = \frac{\omega}{2\pi} = \frac{1}{T} = \text{number of waves per unit time}$$

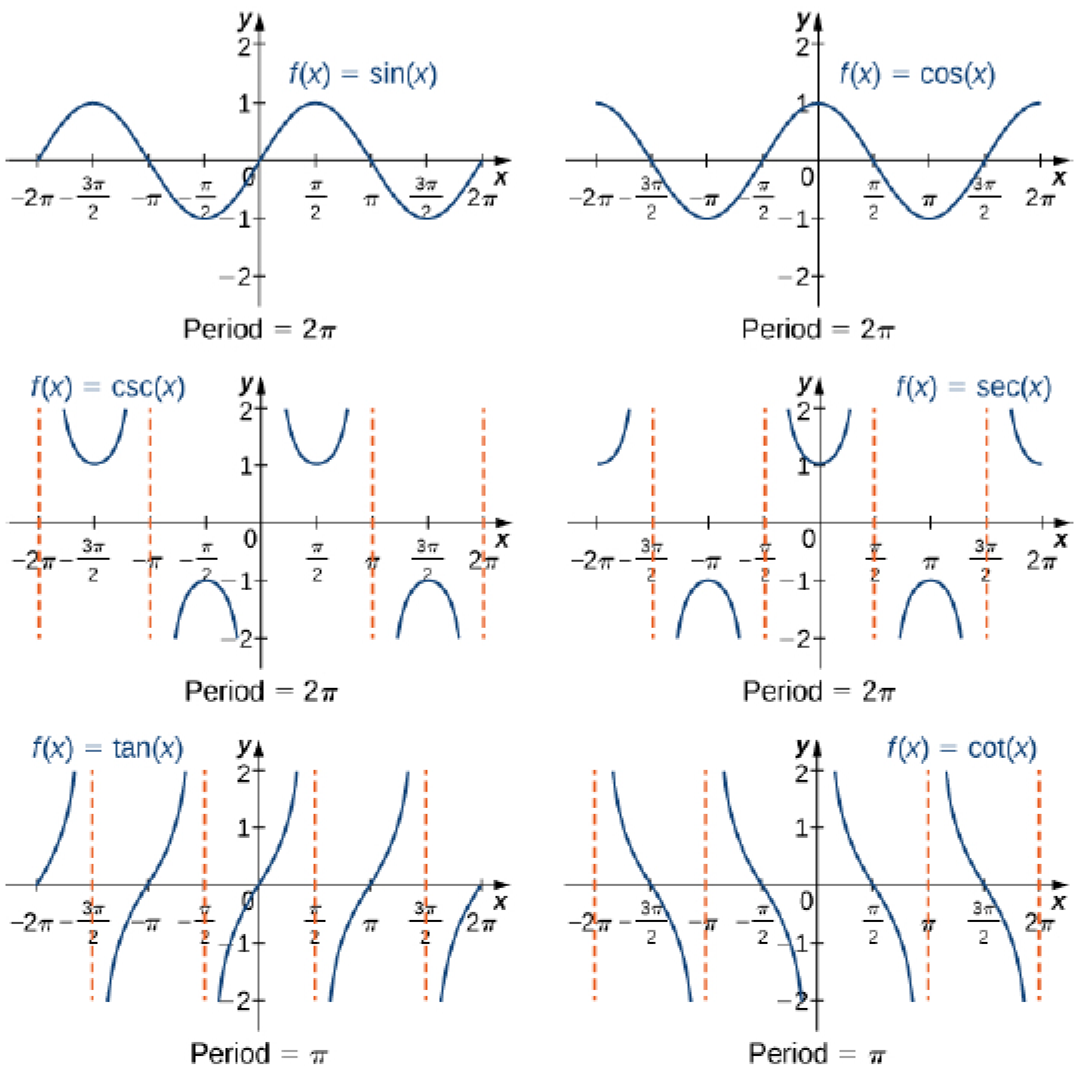
angular frequency, $f_{ang} = \frac{2\pi}{T} = \omega =$ number of waves in a 2π interval (useful if viewing t as an angle in radians).

Sometimes the independent variable is space x e.g. $f(x) = A \sin kx$ and we have;

$$\textbf{wavelength}, \lambda = \frac{2\pi}{k} = \text{spatial extent of one wave}$$

$$\textbf{wavenumber}, \frac{1}{\lambda} = \frac{k}{2\pi} = \text{number of waves in a unit length}$$

$$\textbf{angular wavenumber}, k = \frac{2\pi}{\lambda} = \text{number of waves in a } 2\pi \text{ distance}$$



In contrast to the infinitely differentiable trig functions above, in applications we often encounter periodic functions that are not continuous (especially at $0, T, 2T, \ldots$) but which are made up of continuous pieces, e.g.;

- the sawtooth, $f(x) = x$ for $0 \leq x < 1$ with period 1
- the square wave, $f(x) = 1$ for $0 \leq x < 1$ and $f(x) = 0$ for $1 \leq x < 2$ with period 2

3.1 Fourier's Theorem

Fourier's theorem states that any square-integrable function1 $f(x)$ which is periodic on the interval $0 < x \leq L$ (meaning $f(x + L) = f(x)$ can be written as;

$$f(x) = a_0 = + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L} x\right) \tag{3.2}$$

with

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n}{L} x\right) dx$$

and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi n}{L} x\right) dx$$

This decomposition is known as a Fourier series. Fourier series are useful for periodic functions or functions on a fixed interval L (like a string). One can do a similar analysis for non-periodic functions or functions on an infinite interval ($L \to \infty$) in which case the decomposition is known as a Fourier transform. We will study Fourier series first.

It is easy to verify these formulas for a_n and b_n. However, for a_0, we just integrate $f(x)$. Since $\cos\left(\frac{2\pi}{L} nx\right)$ goes through n cycles of the complete cosine curve as x goes from 0 to L, we have;

$$\int_0^L \cos\left(\frac{2\pi n}{L} x\right) dx = 0, \ \ n > 0 \tag{3.3}$$

Similarly;

$$\int_0^L \sin\left(\frac{2\pi n}{L} x\right) dx = 0, \ \ n > 0 \tag{3.4}$$

Thus,

$$\begin{aligned} \int_0^L f(x) dx &= a_0 \int_0^L dx + \sum_{n=1}^{\infty} a_n \int_0^L \cos\left(\frac{2\pi n}{L} x\right) dx + \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{2\pi n}{L} x\right) dx \\ &= a_0 L \end{aligned} \tag{3.5}$$

or,

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \tag{3.6}$$

Next, for a_n, we can use the cosine sum formula to write;

$$\int_0^L \cos\left(\frac{2\pi n}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)dx = \int_0^L \left[\frac{1}{2}\cos\left(\frac{n+m}{L}2\pi x\right) + \frac{1}{2}\cos\left(\frac{n-m}{L}2\pi x\right)\right]dx \qquad (3.7)$$

Now again, we have that these integrals all vanish over an integer number of periods of the cosine curve. The only way this wouldn't vanish is if $n - m = 0$. So, we have for $n > 0$;

$$\int_0^L \cos\left(\frac{2\pi n}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)dx = \frac{1}{2}\delta_{mn}\int_0^L dx = \frac{L}{2}\delta_{mn} \qquad (3.8)$$

where δ_{mn} is the Kronecker δ-function;

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \qquad (3.9)$$

Similarly,

$$\int_0^L \cos\left(\frac{2\pi m}{L}x\right)\sin\left(\frac{2\pi n}{L}x\right)dx = 0 \qquad (3.10)$$

and,

$$\int_0^L \sin\left(\frac{2\pi m}{L}x\right)\sin\left(\frac{2\pi n}{L}x\right)dx = \frac{L}{2}\delta_{mn} \qquad (3.11)$$

Thus, for $n > 0$;

$$\int_0^L f(x)\cos\left(\frac{2\pi n}{L}x\right)dx = \int_0^L \left[a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi m}{L}x\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{2\pi m}{L}x\right)\right]\cos\left(\frac{2\pi n}{L}x\right)dx \qquad (3.12)$$

$$= \frac{L}{2}\sum_{m=1}^{\infty} a_m \delta_{mn} = \frac{L}{2}a_n \qquad (3.13)$$

So,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n}{L}x\right)dx \qquad (3.14)$$

Similarly, if we multiply both sides of Equation 3.5 by $\sin mx$ and integrate, we get;

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \qquad (3.15)$$

Example 3.1: Find the Fourier series for the **sawtooth function**: $f(x) = x$ for $0 < x \leq 1$.

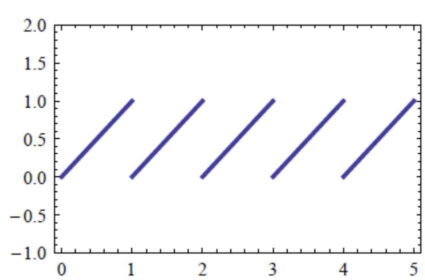


Fig. 3.1: Sawtooth function

This function is clearly periodic. It is equal to $f(x) = x$ on the interval $0 < x \leq 1$. Thus, we can compute the Fourier series with $L = 1$. We get;

$$a_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\int_0^1 xdx = \frac{1}{2}$$

Next,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n}{L}x\right)dx = 2\int_0^1 x\cos 2(\pi n x)$$

This can be done by integration by parts;

$$a_n = 2\frac{x}{2\pi n}\sin(2\pi n x)|_0^1 - 2\int_0^1 \sin(2\pi n x)dx = 0$$

Finally,

$$b_n = 2\int_0^1 x\sin(2\pi n x)dx = -2\frac{x}{2\pi n}\cos(2\pi n x)|_0^1 + 2\int_0^1 \cos(2\pi n x)\,dx = -\frac{1}{\pi n}$$

Thus, we have;

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{\pi n}\sin\left(\frac{2\pi n x}{L}\right)$$

Let's look at how well the series approximates the function when including various terms. Taking 0, 1 and 2 terms in the sum gives;

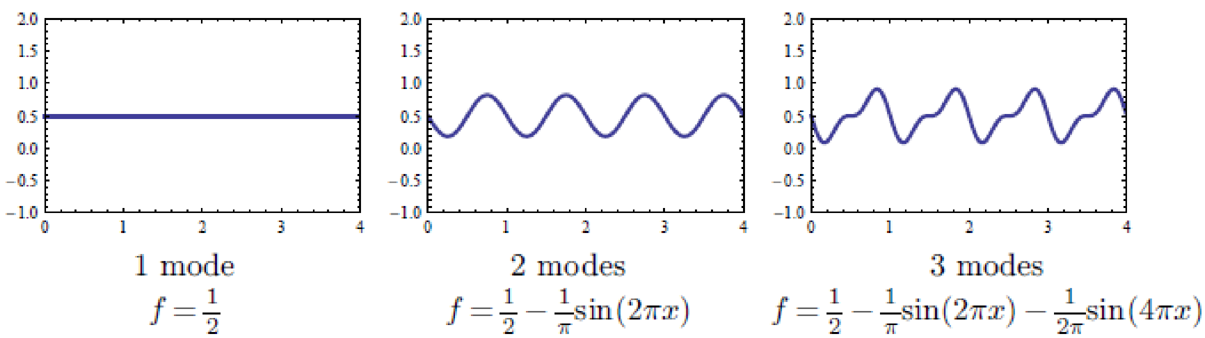


Fig. 3.2: Approximations to the sawtooth function

Already at 3 modes, it's looking reasonable. For 5, 10 and 100 modes we find;

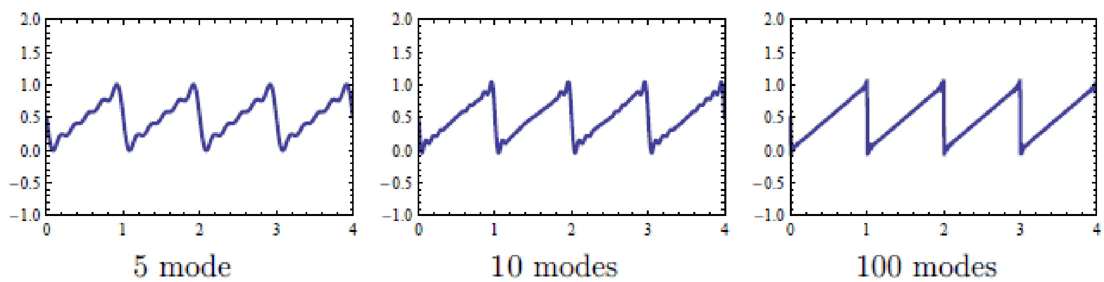
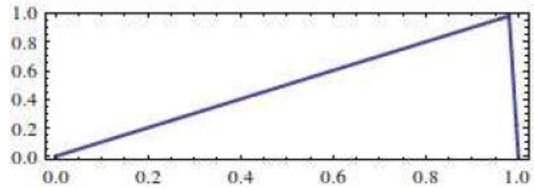


Fig. 3.3: More approximations to the sawtooth function

Observe that for 10 modes we find excellent agreement.

Example 3.2: Plucking a string

Let's apply the Fourier decomposition we worked out to plucking a string. Suppose we pluck a string by pulling up one end:



What happens to the string? To find out, let us do a Fourier decomposition of the x-dependence of the pluck. We start by writing;

$$A(x,t) = \sum_{n=0}^{\infty}\left[a_n \cos\left(\frac{2n\pi}{L}x\right)\cos(\omega_n t) + b_n \sin\left(\frac{2n\pi}{L}x\right)\cos(\omega_n t)\right], \quad \omega_n = \frac{2n\pi}{L}v \tag{3.16}$$

Here v is the speed of sound in the string. For a given wavenumber, $k_n = \frac{2n\pi}{L}$, we know that $\omega_n = k_n v$ to satisfy the wave equation. We could also have included components with $\sin(\omega_n t)$; however since the string starts off at rest (so that $\partial_t A(x,t) = 0$), then the coefficients of the $\sin(\omega_n t)$ functions must all vanish.

At time $t = 0$, the amplitude is;

$$A(x,0) = \sum_{n=0}^{\infty}\left[a_n \cos\left(\frac{2n\pi}{L}x\right) + b_n \sin\left(\frac{2n\pi}{L}x\right)\right] \tag{3.17}$$

This is just the Fourier decomposition of the function described by our pluck shape. If we approximate the pluck as the sawtooth function from the previous section, then we already know that;

$$a_n = 0, \quad b_n = -\frac{1}{\pi n} \tag{3.18}$$

So that, setting $L = 1$;

$$A(x,t) = \sum_{n=1}^{\infty} -\frac{1}{\pi n}\sin(2\pi x)\cos(2\pi v t) \tag{3.19}$$

This gives the motion of the string for all time.

The relative amplitudes of each mode are;

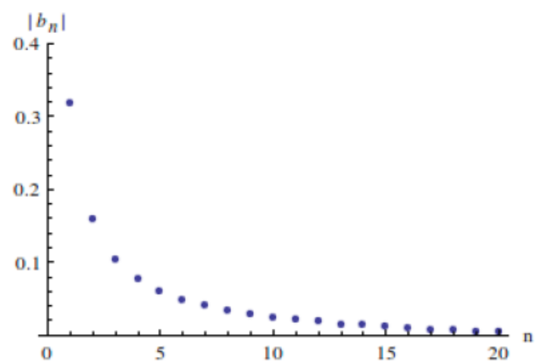


Fig. 3.4: Amplitudes of the relative harmonics of a string plucked with a sawtooth plucking.

The $n = 1$ mode is the largest. This is the fundamental frequency of the string. Thus the sound that comes out of the string will be mostly this frequency: $\omega_1 = \frac{2\pi}{L} v$. The modes with $n > 1$ are the **harmonics**. Harmonics have frequencies which are integer multiples of the fundamental.

Definition 3.3 Functions with period 2π: Let f be a piecewise continuous function on $[-\pi, \pi]$. Then the Fourier series of f is the series;

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0 and the coefficients a_n and b_n in this series are defined by;

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx \tag{3.20}$$

This is an alternative definition to when $f(x)$ is periodic on the interval $0 < x \leq L$ (meaning $f(x + L) = f(x)$. Here $f(x + 2\pi) = f(x)$ on $-\pi < x < \pi$.

Example 3.4 Find the Fourier coefficients and Fourier series of the square-wave function f defined by;

$$f(x) = \begin{cases} 0 & \text{if} -\pi \leq x < 0 \\ 1 & \text{if} \quad 0 \leq x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x)$$

so f is periodic with period 2π.

Using the formulas for the Fourier coefficients in definition in equation (3.20), we have;

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{0} 0 dx + \frac{1}{2\pi} \int_{0}^{\pi} 1 dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

and for $n \geq 1$;

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 dx + \frac{1}{\pi} \int_{0}^{\pi} \cos nx \, dx = 0$$

and;

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 dx + \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

The Fourier series of f is therefore;

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \cdots$$

Since odd integers can be written as $n = 2k - 1$, where k is an integer, we can write the Fourier series in sigma notation as;

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)x$$

Functions with period $2L$

If a function f has period other than 2π, we can find its Fourier series by making a change of variable. Suppose $f(x)$ has period $2L$, that is;

$$f(x + 2L) = f(x) \quad \forall \, x$$

If we let $t = \frac{\pi x}{L}$ and;

$$g(t) = f(x) = f\left(\frac{Lt}{\pi}\right)$$

then, as you can verify, g has period 2π and $x = \pm L$ corresponds to $t = \pm\pi$. The Fourier series of g is;

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where;

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)dt, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt \; dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt \; dt$$

If we now substitute for $x = \frac{Lt}{\pi}$, then $t = \frac{\pi x}{L}$, and $dt = \frac{\pi}{L}dx$.

Definition 3.5 Let f be a piecewise continuous function on $[-L, L]$. Then the Fourier series of f is the series;

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

and, for $n \geq 1$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Example 3.6 Find the Fourier series of the triangular wave function defined by;

$$f(x) = |x| \;\; for \; -1 \leq x \leq 1 \;\; and \;\; f(x+2) = f(x) \;\; \forall x$$

For which values of x is $f(x)$ equal to the sum of its Fourier series?

We find the Fourier coefficients by putting $L = 1$ in Definition 3.5:

$$a_0 = \frac{1}{2}\int_{-1}^{1}|x|dx = \frac{1}{2}\int_{-1}^{0}(-x)dx + \frac{1}{2}\int_{0}^{1} x dx = \frac{1}{2}$$

and for $n \geq 1$;

$$a_n = \int_{-1}^{1}|x|\cos(n\pi x)dx = 2\int_{0}^{1} x\cos(n\pi x)dx = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore, the Fourier series is;

$$\frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x) - \frac{4}{9\pi^2}\cos(3\pi x) - \frac{4}{25\pi^2}\cos(5\pi x) - \cdots$$

$$= \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x)$$

The triangular wave function is continuous everywhere and so, accordingly, we have;

$$f(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x) \quad \forall x$$

In particular,

$$|x| = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x) \quad -1 \leq x \leq 1$$

Example 3.7 Exponentials

Fourier series decompositions are even easier with complex numbers. There we can replace the sines and cosines by exponentials. The series is;

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}$$

where

$$c_n = \frac{1}{L}\int_0^L f(x) e^{-inx\frac{2\pi}{L}} dx$$

To check this, we substitute in;

$$\int_0^L f(x) e^{-inx\frac{2\pi}{L}} dx = \sum_{m=-\infty}^{\infty} c_m \int_0^L e^{-imx\frac{2\pi}{L}} e^{-inx\frac{2\pi}{L}} dx = \sum_{m=\infty}^{\infty} c_m \int_0^L e^{i(m-n)x\frac{2\pi}{L}} dx$$

If $n \neq m$, then;

$$\int_0^L e^{i(m-n)x\frac{2\pi}{L}} dx = \frac{L}{2\pi}\frac{1}{m-n} e^{i(m-n)\frac{2\pi}{L}x}\Big|_0^L$$

$$= \frac{L}{2\pi}\frac{1}{m-n}\left[e^{i2\pi(m-n)} - 1\right] = 0$$

If $m = n$, then the integral is just;

$$\int_0^L dx = L; \quad \text{thus,} \quad \int_0^L e^{i(m-n)x\frac{2\pi}{L}} dx = L\delta_{mn}; \quad \text{and so,} \quad \int_0^L f(x) e^{-inx\frac{2\pi}{L}} dx = Lc_n$$

If $f(x)$ is real, then;

$$f(x) = \sum_{n=-\infty}^{\infty} \mathrm{Re}(c_n + c_{-n}) \cos\left(\frac{2\pi nx}{L}\right) + \mathrm{Im}(c_{-n} + c_n) \sin\left(\frac{2\pi nx}{L}\right)$$

So, $a_n = \mathrm{Re}(c_n + c_{-n})$ and $b_n = Im(c_{-n} + c_n)$. Thus, the exponential series contains all the information in both the sine and cosine series in an efficient form.

4 Fourier Transform

The origin of the Fourier transform can be traced back to the work of Jean-Baptiste Joseph Fourier, a French mathematician and physicist, in the early 19th century. Fourier was interested in solving the heat equation, which describes how heat flows in a solid material over time. Fourier recognized that any periodic function could be represented as an infinite sum of sine and cosine functions of different frequencies. He proposed that by decomposing a function into its constituent sinusoidal components, it would be possible to analyze and understand its behavior more easily.

Fourier's key insight was that even non-periodic functions could be represented as an integral (continuous case) or a sum (discrete case) of sinusoidal functions of different frequencies. He developed a mathematical technique to express a function in terms of its frequency content, which we now refer to as the Fourier transform. Fourier published his work on the theory of heat conduction and the decomposition of functions in his monumental treatise, "Théorie analytique de la chaleur" (Analytical Theory of Heat), in 1822. In this work, he laid out the principles and mathematical foundations of what we now know as the Fourier series and Fourier transform.

Fourier's ideas initially faced some resistance and skepticism from the mathematical community. However, over time, his work gained recognition and became widely accepted. The Fourier transform has since become a fundamental tool in many branches of science and engineering, revolutionizing fields such as signal processing, image analysis, quantum mechanics, and many more.

Today, the Fourier transform is an essential mathematical technique used in various applications and is taught in mathematics, physics, engineering, and other related disciplines. It has paved the way for numerous advancements and discoveries, and its impact on modern science and technology cannot be overstated.

4.1 Applications of Fourier Transform

The Fourier transform has a wide range of applications across various fields. Some of the key applications include:

1) **Signal Processing:** The Fourier transform plays a crucial role in signal processing. It allows the analysis, manipulation, and filtering of signals in the frequency domain. Applications include audio and speech processing, image and video compression, noise reduction, and equalization.
2) **Communication Systems:** The Fourier transform is used in the design and analysis of communication systems. It helps in modulating and demodulating signals, multiplexing and demultiplexing multiple signals, and understanding the frequency characteristics of channels.
3) **Image and Video Processing:** The Fourier transform is used extensively in image and video processing applications. It enables tasks such as image enhancement, compression, denoising, and feature extraction. Techniques like the Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT) are commonly used in these applications.
4) **Spectral Analysis:** The Fourier transform is employed to analyze the frequency content of signals in various domains, including physics, astronomy, and geology. It allows the identification and characterization of different frequency components in a signal, aiding in phenomena analysis and pattern recognition.
5) **Quantum Mechanics:** In quantum mechanics, the Fourier transform is used to analyze the wave functions of quantum systems. It helps in understanding the momentum and position space representations of quantum particles, as described by the Heisenberg uncertainty principle.

6) **Medical Imaging:** The Fourier transform is utilized in medical imaging techniques such as magnetic resonance imaging (MRI) and computed tomography (CT). It allows the conversion of raw data into image space, enabling visualization and analysis of anatomical structures and abnormalities.
7) **Speech Recognition and Synthesis:** The Fourier transform is used in speech recognition systems to extract the frequency content of speech signals for feature extraction and classification. It is also used in speech synthesis techniques like vocoders and speech compression algorithms.
8) **Astrophysics and Cosmology:** The Fourier transform is employed in analyzing astronomical data and studying the properties of celestial objects. It helps in determining the frequency spectra of cosmic radiation, understanding the dynamics of galaxies, and studying the cosmic microwave background radiation.

These are just a few examples of the diverse applications of the Fourier transform. Its versatility and effectiveness in analyzing and manipulating signals in the frequency domain have made it an indispensable tool in numerous scientific, engineering, and technological fields.

4.2 Formulas of Fourier Transform

The Fourier transform is a generalization of the complex Fourier series (see example 3.7 above). The complex Fourier Series is an expansion of a periodic function (periodic in the interval $[-L/2, L/2]$) in terms of an infinite sum of complex exponential:

$$\sum_{n=-\infty}^{\infty} A_n e^{i2\pi nx/L} \tag{4.1}$$

Where the coefficients A_n are;

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-i2\pi nx/L} dx \tag{4.2}$$

Note that this expansion of a periodic function is equivalent to using the exponential functions $u(x) = e^{i2\pi nx/L}$ as a *basis* for the function vector space of periodic functions. The coefficient of each 'vector' in the basis are given by the coefficient A_n. Accordingly, we can interpret equation (4.2) as the *inner product* $\langle u_n(x)|f(x)\rangle$.

In the limit as $L \to \infty$ the sum over n becomes an integral. The discrete coefficients A_n are replaced by the continuous function $F(k)dk$ where $k = n/L$. Then in the limit ($L \to \infty$) the equations defining the Fourier series become;

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{i2\pi xk} dk \tag{4.3}$$

and

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-i\pi kx} dx \tag{4.4}$$

Here,

$$F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx \tag{4.5}$$

is called the *forward* Fourier transform, and;

$$f(x) = F_k^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k)e^{i2\pi xk}dk$$

is called the *inverse* Fourier transform.

The notation $F_x[f(x)](k)$ is common but $\bar{f}(k)$ and $\hat{f}(x)$ are sometimes also used to denote the Fourier transform.

In physics we often write the transform in terms of angular frequency $\omega = 2\pi v$ instead of the oscillation frequency v (thus for example were place $2\pi k \to k$). To maintain the symmetry between the forward and inverse transforms, we will then adopt the convention;

$$F(k) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

and

$$f(x) = F^{-1}[F(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ixk}dk$$

4.3 Properties of Fourier Transforms

If f and g are integrable, then the following holds:

1) $F[f(x-a)] = e^{ika}\hat{f}(k)$
2) $F\left[f(x)e^{ibx}\right] = \hat{f}(k-b)$
3) $F[f(\lambda x)] = |\lambda|^{-1}\hat{f}(\lambda^{-1}k)$
4) $F\left[\hat{f}(x)\right] = f(-k)$
5) $F[f'(x)] = ik\hat{f}(k)$
6) $F[xf(x)] = i\hat{f}'(k)$
7) $F[(f * g)(x)] = \sqrt{2\pi}\hat{f}(k)g(k)$
8) $F[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(k)$

We shall demonstrate how to prove some of these properties while other proofs follow in similar manner:

Proof: (Property 1-4)

These properties follow immediately from a change in variables. For example, property 1 follows thus;

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-ik(y+a)}dy \qquad (y = x - a)$$

$$= \frac{e^{-ika}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-iky}dy = e^{-ika}\hat{f}(k)$$

(Property 5-6) These properties follow immediately by interchanging differentiation and integration. For example, if we write $f(x)$ as the inverse Fourier transform of $\hat{f}$, then;

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

then computing the derivative implies that;

$$f'(x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ik\hat{f}(k)e^{ikx}dk = F^{-1}[ikf(k)]$$

which implies that;

$$F[f'(x)] = ik\hat{f}(k)$$

We are able to pass the derivative inside the integral by Leibniz's rule provided that $ik\hat{f}(k)e^{ikx}$ is integrable. This integrability condition is implicitly satisfied because we assumed that the functions in the theorem have convergent Fourier and inverse Fourier transforms.

We shall however omit the proof of properties 7-8 and interested students can read it up from appropriate academic material.

4.4 Relationship Between Fourier Transforms and Inverse Fourier Transforms

1) $F[f(x-a)] = e^{ika}\hat{f}(k)$
2) $F^{-1}\left[F[f(x)]\right] = f(x)$
3) $F\left[F[f(x)]\right] = f(-x)$
4) $F^{-1}\left[F^{-1}[f(x)]\right] = f(-x)$

Example 4.1 Find the Fourier transform of;

$$f(x) = e^{-a|x|}, \quad a > 0$$

Solution: This can be computed directly. We split the region of integration;

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-a|x|}dx = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{-ikx+ax}dx + \int_{0}^{\infty} e^{-ikx-ax}dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ikx+ax}}{a-ik}\bigg|_{x=-\infty}^{x=0} + \frac{e^{-ikx-ax}}{-a-ik}\bigg|_{x=0}^{x=\infty}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a-ik} + \frac{1}{a+ik}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2+k^2}\right)$$

Example 4.2 Find the Fourier transform of;

$$f(x) = e^{-\frac{x^2}{2}}$$

Solution: This can be computed directly. We do a complex change of variables;

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2}dx \quad \text{(complete the square)}$$

$$= \frac{1}{\sqrt{\pi}}e^{-\frac{k^2}{2}}\int_{-\infty}^{\infty} e^{-z^2}dz$$

$$= e^{-\frac{k^2}{2}} \qquad \left(\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}\right)$$

Remark: The imaginary change of variables $z = \frac{1}{\sqrt{2}}(x+ik)$ can be justified using complex analysis;

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x+ik)^2}dx = \sqrt{2}\int_{\mathbb{R}+\frac{ik}{\sqrt{2}}} e^{-z^2}dz = \sqrt{2}\int_{-\infty}^{\infty} e^{-z^2}dz$$

Example 4.3 Find the Fourier transform of;

$$f(x) = xe^{-a\frac{x^2}{2}}, \quad a > 0$$

Solution: Instead of computing it directly, we start from the Fourier transform of the Gaussian;

$$g(x) = e^{-\frac{x^2}{2}} \quad \Rightarrow \quad \hat{g}(k) = e^{-\frac{k^2}{2}}$$

Since;

$$f(x) = xe^{-a\frac{x^2}{2}} = x \cdot g(\sqrt{a}x)$$

the properties of Fourier transform imply that;

$$\begin{aligned} \hat{f}(k) &= i\frac{d}{dk}F\left[g\left(\sqrt{a}x\right)\right](k), \quad (xf(x) \mapsto i\hat{f}'(k)) \\ &= i\frac{d}{dk}\left(\frac{1}{\sqrt{a}}e^{-\frac{k^2}{2a}}\right), \qquad (f(\lambda x) \mapsto |\lambda|^{-1}\hat{f}(\lambda^{-1}k)) \\ &= -ika^{-\frac{3}{2}}e^{-\frac{k^2}{2a}} \end{aligned}$$

4.5 Fourier Cosine Transform (FCT)

The infinite Fourier cosine transform of $f(x)$ is defined by;

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\, dx$$

and the inverse Fourier cosine transform of $F_c[f(x)]$ is define by;

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos kx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty F(k)\cos kx\, dk$$

Where $F_c[f(x)]$ and $F^{-1}F_c[f(x)]$ are called Fourier cosine pairs.

4.6 Fourier Sine Transform (FST)

The infinite Fourier sine transform of $f(x)$ is defined by;

$$F_s[f(x)] = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\, dx$$

The inverse Fourier sine transform of $F_s[f(x)] = F_s(k)$ is defined by;

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(k)\sin kx\, dk$$

and $F_s[f(x)]$ and $F^{-1}F_s[f(x)]$ are called Fourier sine transform pairs.

Example 4.4: Find the Fourier cosine transform of;

$$f(x) = \begin{cases} \cos x & \text{if } \ 0 < x < a \\ 0 & \text{if } \ x \geq a \end{cases}$$

Solution:

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx$$

Thus, for $0 < x < a$,

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos kx \, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^a [\cos(1+k)x + \cos(1-k)x] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1+k)x}{1+k} + \frac{\sin(1-k)x}{1-k} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1+k)a}{1+k} + \frac{\sin(1-k)a}{1-k} - (0+0) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1+k)a}{1+k} + \frac{\sin(1-k)a}{1-k} \right] \quad \text{provided} \quad k \neq \pm 1$$

Example 4.5: Find the Fourier cosine transform of;

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

Solution:

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos kx \, dx + \int_1^2 (2-x) \cos kx \, dx + \int_2^\infty 0. \cos kx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[x \frac{\sin kx}{k} - 1 \left(-\frac{\cos kx}{k^2} \right) \right]_0^1 + \left[(2-x) \frac{\sin kx}{k} - (-1) \left(-\frac{\cos kx}{k^2} \right) \right]_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin k}{k} + \frac{\cos k}{k^2} - \frac{1}{k^2} - \frac{\cos 2k}{k^2} - \frac{\sin k}{k} + \frac{\cos k}{k^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\cos k - \cos(2k - 1)}{k^2} \right]$$

Example 4.6: Find the Fourier sine transform for (i) $\frac{1}{x}$, (ii) $2e^{-3x} + 3e^{-2x}$

Solution:

(i)

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin kx\, dx = \sqrt{\frac{2}{\pi}} * \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

(ii)

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty (2e^{-3x} + 3e^{-2x})\sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty (2e^{-3x}\cos kx + 3e^{-2x}\sin kx)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2e^{-3x}}{9+k^2}(-3\sin kx - k\cos kx)\right]_0^\infty + \sqrt{\frac{2}{\pi}}\left[\frac{3e^{-2x}}{4+k^2}(-2\sin kx - k\cos kx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(0 + \frac{2k}{9+k^2}\right) + \sqrt{\frac{2}{\pi}}\left(0 + \frac{3k}{4+k^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{2k}{9+k^2} + \frac{3k}{4+k^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{5k^3 + 35k}{(9+k^2)(4+k^2)}\right)$$

Assignment

1) Find the Fourier transform of;

$$f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and hence evaluate;

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx$$

2) Find the Fourier cosine transform of;

$$f(x) = \left(\frac{1}{1+x^2}\right)$$